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Quarks in the Skyrme-'t Hooft-Witten Model

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Abstract

The three-flavor Skyrme-'t Hooft-Witten model is interpreted in terms of a quark-like substructure, leading to a new model of explicitly confined color-free "quarks" reminiscent of Gell-Mann's original pre-color quarks, but with unexpected and significant differences.

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The standard theory for strong interaction physics is, by consensus, QCD, but the inherent difficulties in applying this theory to long-range low-energy phenomena (meson-nuclear physics, say) have so far been insuperable. In the mid-seventies 't Hooft[1] demonstrated that the gauged color group $SU(N_c)$, with $N_c \to \infty$, provided an approximate approach to strong interactions. In this limit, mesons have masses that scale as $(N_c)^0$ and the meson resonances are narrow with widths that scale as $(N_c)^{-1}$. In contrast, baryons (which contain N_c quarks) have masses that scale as $(N_c)^1$ with sizes and shapes that have an N_c independent limit. Witten[2] recognized that, for baryons, these are characteristics of a soliton, reviving an earlier (topological) strong interaction model of Skyrme[3].

The Skyrme-'t Hooft-Witten (S'tW) model of baryons results in a classical soliton solution of the nonlinear chiral $SU(N_f) \times SU(N_f)$ model, from which the quantal baryonic states can be projected. The crucial new ingredient in the S'tW model, due to Witten, is the *anomaly term*. The implications of this anomaly term are truly remarkable, and this term accounts for the impressive qualitative (structural) agreements of the model with observation. For two-flavors the anomaly does not exist and the S'tW model is topologically trivial, essentially equivalent, in fact, to the old strong-coupling spin-isospin model. Deeper insight into the structure of the two-flavor S'tW model came from the quark hedgehog (large-N_c quark) analysis[4]. This analysis, combined with **K**-symmetry, as discussed by Mattis and Braaten[5], led to predictions for *two-flavor* meson-baryon scattering in surprisingly good agreement with experiment.

The situation for the three-flavor S'tW model - - as a direct consequence of the anomaly term - - is totally different. Here the analysis in terms of large-N_c quarks is - - as we will prove - -

incorrect in the literature. Part of the problem is the understandable, but unfortunate, confusion caused by denoting two distinct concepts by the same symbol. The S'tW model, when analyzed and interpreted in the way we propose, can be seen as a model based on *quark-solitons*, which are generalized (topological) "quarks" strongly reminiscent of Gell-Mann's original (pre-color) "mathematical" quarks[6] - - though not without some unusual features of their own, as we shall show.

The three-flavor S'tW model is defined[2] by the Poincaré invariant action:

$$S = \int d^4 x L + n\Gamma, (n \in Z), \tag{1}$$

where Γ is the anomaly and L is the Skyrme Lagrangian,

$$L = \frac{F_\pi^2}{16} Tr\left\{[\partial_\mu U \partial^\mu U]\right\} - \frac{1}{32e^2} Tr\left\{[(\partial_\mu U)U, (\partial_v U)U]^2\right\}, \tag{2}$$

with $U(x_\mu) \in SU(3),\ F_\pi \cong 186 Mev$, and e is a dimensionless constant. The anomaly[2] cannot be written as an integral over space-time, but appears in the form:

$$\Gamma = \frac{1}{240\pi^2} \int d\Sigma^{ijklm} Tr(V_i V_j V_k V_l V_m) \tag{3}$$

with $V_j \equiv -U^{-1}\partial_j U, U \in SU(3)$, and $d\Sigma^{ijklm}$ a volume element in an extended five-dimensional space; the boundary of the integration region is compactified space-time.

Topological considerations enter as follows: for a given time t, the matrix $U(t, x)$ is a mapping from $\boldsymbol{R}^3$ into $SU(3)$. The proper boundary conditions add the point at infinity to three-space compactifying it to S^3. As is familiar from Witten's work,[2] the equivalence classes of all such maps are classified by the homotopy group $\pi_3(SU(3)) = Z$, (the integral baryon number B).

For three-flavors the lowest energy B=1 soliton of the S'tW model is the 3×3 matrix:

$$\Sigma(r) = \begin{pmatrix} \exp(\frac{2i}{F_\pi}\tau \cdot \hat{r} F(r)) & 0 \\ 0 & 1 \end{pmatrix}, \tag{4}$$

where τ denotes the 2×2 isospin matrices and $F(r)$ is determined from the Euler-Lagrange equations. The soliton Σ has the symmetry $SU(3)_{flavor} \times SU(2)_{spin}$, where $SU(3)_{flavor}$ is realized by the adjoint action and $SU(2)_{spin}$ by transformations on $\mathbf{r}$ (generators $\mathbf{J}$. Eq.(4) shows a special "**K**-symmetry" in that Σ is invariant under combined isospin-spin rotations: $(\mathbf{I} + \mathbf{J})(\Sigma) = 0$, and, moreover, the $SU(3)_{flavor}$ hypercharge generator Y also leaves Σ invariant: $[Y, \Sigma] = 0$.

The quantal eigenstates of the S'tW model are projected from the soliton Σ, and are *monopolar harmonics*[7], sections of a fiber bundle over the coset manifold $SU3/U1$. These monopolar harmonics are specializations of the $SU3$ rotation matrices over eight angles $(\alpha_1, i = 1 \ldots 8)$, with the angle a_8 determined by the sectional map. The $SU(3) \times SU(3)$ symmetry of the $SU(3)$ rotation matrices (generated by left and right actions on the $SU(3)$ manifold) reduces to the symmetry $SU(3)_{flavor} \times SU(2)_{spin}$ for the monopolar harmonics, with the anomalous right action hypercharge $Y_R \to N_c B/3$.

Can one construct a large-N_c quark model unitarily equivalent to this S'tW three- flavor model? Manohar[4] was the first to attempt this and his model was based on the hedgehog quark:

$$|\Sigma_q\rangle \equiv (|u\rangle\otimes|\downarrow\rangle - |d\rangle\otimes|\uparrow\rangle) \tag{5}$$

where (as in the two-flavor case) $\boldsymbol{K}\,|\Sigma_q\rangle = 0$, where $\boldsymbol{K} = \boldsymbol{I} + \boldsymbol{J}$. What distinguishes this ket vector, $|\Sigma_q\rangle$, from the two-flavor case is the set of allowed operations: for the three-flavor case, one allows arbitrary $SU(3)_{flavor}$ and $SU(2)_{spin}$ transformations to act on $|\Sigma_q\rangle$. Transforming $|\Sigma_q\rangle$ by $R(g)$, where $g \in SU(3)_{flavor}$, carries $|\Sigma_q\rangle$ to sufficiently many independent states to enable one to determine the six basis states of the defining irrep of $SU(3)_{flavor} \times SU(2)_{spin}$.

Consider now the anti-symmetrized color singlet N_c-hedgehog quark state:

$$|\Sigma_q\rangle_{N_c} \equiv \underbrace{(|\Sigma_q\rangle\otimes|\Sigma_q\rangle\ldots\otimes|\Sigma_q\rangle)}_{N_c times}\chi_{color\ \ singlet}, \tag{6}$$

If we perform the transformation: $|\Sigma_q\rangle \rightarrow R(g)\,|\Sigma_q\rangle$, $g \in SU(3)_{flavor}$ this induces a general transformation on $|\Sigma_q\rangle_{N_c}$. The states spanned by $\left\{R(g)\,|\Sigma_q\rangle_{N_c}, g \in SU(3)\right\}$ are precisely the states of the totally symmetric irrep $[N_c\dot{0}]$ in $SU(6)$. The $SU(3) \times SU(2)$ structure of the irrep $[N_c\dot{0}]$ in $SU(6)$ is well known[8]: Every irrep $[m_{13}m_{23}0] \times [m_{13}m_{23}]$ in $SU(3) \times SU(2)$ occurs once and only once, subject to lexicality and the constraint that $m_{13} + m_{23} = N_c$.

To compare the baryonic states in this large N_c quark model with the baryonic states of the three-flavor S'tW model we see that we must choose $N_c = 3k$, with k an integer. This comparison shows that there are serious discrepancies. In fact *only for $N_c = 3$ are the states the same.* One finds that: (a) the spins in Manohar's model for the baryon tower are *not all half-integral.* (b) the *multiplicity* of the $SU(3)_{flavor}$ multiplets in the two systems is not the same, and (c) unlike the two-flavor hedgehog quark model, the multiplets for a given $N_c(= 3k)$ do not contain the multiplets for lower $N_c(= 3(k-1))$. *This means that the physically important lowest multiplets in the tower will be obtained only with $N_c = 3$, which prevents using the large-N_c limit.*

To resolve these discrepancies, we remark that the proper three-flavor hedgehog quark is not defined by (6) but by the *local* [100] monopolar harmonic, which does not exist globally, since such an object is *forbidden topologically.* (More precisely these objects exist locally as confined triples). Let us assume that this [100] state does exist locally; what would it look like? Written as a matrix it would appear trivial, simply the 3×3 unit matrix. But recall that $SU(3)$ rotational wave functions realize the symmetry $SU(3)_{left} \times SU(3)_{right}$ in which (conventionally) the left generators *obey time-reversed commutation rules.*[9] Re-writing the 3×3 unit matrix to accord with this basis, we find that

$$|\Sigma_0\rangle = (|u\rangle\otimes|\downarrow\rangle - |d\rangle\otimes|\uparrow\rangle + |s\rangle\otimes|\rightarrow\rangle)\,. \tag{7}$$

where $|\rightarrow\rangle$ denotes a new "sidewise", or spin 0, component. Clearly the state $|\Sigma_0\rangle$ is invariant under an *octet* of $SU(3)\boldsymbol{K}$-symmetry generators.

We must justify this radical step impled by (7), but before we do so let us remark that *all of the difficulties noted above are resolved by this change.*

At first glance the introduction of a scalar ($S = 0$) "quark" is absurd; there is no experimental evidence for such an object. But before we dismiss this idea (which was, after all, abstracted from the S'tW model) let us be more careful and examine the right hypercharge (the baryon number in the model). We find $B = -2/3$. *Thus the new* $S = 0, Y_R = B = -\frac{2}{3}$ *state is an anti di-quark.* There is credible evidence that the di-quark[6] exists.[10]

We conclude that the adjunction of the anti-diquark to define $|\Sigma_0\rangle$ is not completely unreasonable.

The resulting large-N_c quark model is now straight-forward. We must use $N_c = 3k$ to agree with the states of the three-flavor S'tW model. The single "quark" basis implied by $|\Sigma_0\rangle$ consists of the nine states in $SU(3)_{flavor} \times SU(3)_{right}$. Thus a baryonic structure composed of N_c such quarks consists of dim $[N_c\dot{0}]_{U9}$ symmetric states in $SU(9)$, since the color state is an anti-symmetric singlet in $SU(N_c)$. It is important to realize that this N_c-quark system consists of "baryons" *with various baryonic charges* (right hypercharge Y_R) since the original "quark" basis itself had two distinct baryonic charges ($\frac{1}{3}$ and $-\frac{2}{3}$). *Thus we must project from this system of* N_c "quarks" *the states of* $Y_R = B = 1$ *to accord with the* $B = 1$ *states of the three-flavor S'tW model.* It is easily demonstrated that this projected $SU(3)_{flavor} \times SU(3)_{right} \times SU(N_c)$ "quark" model for $N_c \to \infty$ is unitarily equivalent to the three-flavor S'tW model since the spectrum of states is *identical*. Accordingly we propose to take this new "quark" model more seriously and examine its implications.

The results we have presented so far agree nicely with the S'tW model, and are, group theoretically, unassailable. Despite this, a more critical analysis shows there is a problem: *following convention, we have used the symbol* N_c *ambiguously, in two distinct, conflicting, ways.* If the anomaly were absent, there would be no ambiguity: one simply takes $SU(3)_{flavor} \times SU(2)_{spin} \times SU(N_c)\ QCD$ in the 't Hooft limit $N_c \to \infty$ obtaining $SU(3)_{flavor} \times SU(2)_{spin}$ symmetry, defined (*a la* Witten) on the manifold $SU(3)/U1$. For clarity, let us henceforth call 't Hooft's parameter $N_{'t}$. Now consider the anomaly. First of all the anomaly *cannot even be defined* until we have constructed the limit $N_{'t} \to \infty$, obtaining a smooth manifold (over $N_{'t}$). Thus the anomaly, which is quantized, introduces a *new integral parameter*, confusingly called N_c again. To avoid confusion let us call this Witten's parameter[11] denoted by N_W.

The limit $N_{'t} \to \infty$ clearly eliminates the parameter $N_{'t}$ from the S'tW model, which means *that in the S'tW model color has been totally confined and disappears from the model.* The anomaly makes a profound difference. Witten's parameter N_W must be 3, which implies that right hypercharge measures the baryon quantum number B. On purely topological grounds, *the anomaly forbids the existence of S'tW states with fractional B (that is, non-zero triality[12]). Triples* of objects, with composite triality zero, are, however, *not* forbidden *if confined within the volume of the Skyrmion* (or to even smaller distances). Thus even though color has disappeared we still have a remnant: quark confinement in triples. (This beautifully illustrates Gell-Mann's intuition that quarks may be fictitious (unobservable) in his classic pre-color paper.[6])

For consistency with both the large-$N_{'t}$ quark model and the admissible quark triples, one must take the $N_{'t} \to \infty$ limit to run over the integers $N_{'t} \equiv 0\ mod\ 3$. Thus we see the true relationship[13] between 't Hooft's parameter $N_{'t}$ in the large-$N_{'t}$ quark model and Witten's anomaly parameter N_W is that $N_{'t} \equiv t \bmod N_W$ with $N_W = 3$. (Here $t = 0, \pm 1$ is the triality of

the $SU(3)_{flavor}$ irrep.)

Accordingly we abstract from the S'tW model the following structure: a quark soliton is a *confined* 'solution' to the S'tW model having $B = \frac{1}{3}$, belonging to the *broken* spectrum generating symmetry $U(9) \subset SU(3)_{flavor} \times SU(3)_{right} \subset SU(3)_{flavor} \times SU(2)_{spin} \times U1_{Y_R}$. Projected onto quantal states, this $B = \frac{1}{3}$ system has an infinite tower of states: $[100]\frac{1}{2}$,$[220]\frac{1}{2}$,$[310]\frac{1}{2}$,$[310]\frac{3}{2}$, $[400]\frac{3}{2}$; .... A finite subset of these soliton quark states can be generated by $N_{\prime t} \equiv 1$ mod 3 confined colorless totally symmetrized light (mass $\sim N_{\prime t}^{-1}$) "quarks", comprising the states: $[100]\frac{1}{2}\ldots, [N_{\prime t}00](\frac{2N_{\prime t}+1}{6})$. The smallest such subset $N_{\prime t} = 3$) are the *six* $B = 1/3$ pre-color Gell-Mann quarks: [100]1/2, and, in addition, the *three* (structurally essential) $B = -\frac{2}{3}$ "quarks" [100]0, that is, nine $[100]_{flavor} \times [100]_{right}$ states in all. Composite projected states of three quark solitons are baryons (B =1).

Quark soliton composites are automatically equipped with a certain form of interaction: it is easily shown[14] that in the limit $N_{\prime t} = \infty$ *all interactions are local in the group (coset) manifold coordinates.* This local interaction predicts three-flavor meson-baryon scattering[15] for $N_{\prime t} = \infty$, generalizing the earlier (two-flavor) results of Mattis-Braaten.[5]

The manifold coordinates (seven dimensional) are dual to the representation-label space (seven dimensional) so that finite $N_{\prime t}$ is to be associated with non-locality which can be modelled by group-theoretically defined tensor operators. Expressed differently the loss of color degrees of freedom by explicit confinement is compensated by the freedom to model, group-theoretically, interactions in composite (large-$N_{\prime t}$) states. The model we propose implies structurally many features previously introduced heuristically. Among these are: suppression of orbital angular momentum (quark-diquark baryonic excited states), explicit ('bag model') quark confinement, massless quarks, zero flavor triality, the 'spin-baryonic charge' rule ($2S \equiv B \; mod \; 2$) and suppression of nuclear 'hidden-color' effects.

The $SU(3)_{color}$ symmetry of QCD is a deep and profound organizing principle in particle physics. It will be interesting to see if soliton quarks, which define a *color-free* approach to long-range, low energy QCD structures, can successfully approximate the real meson-nuclear world.

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[11] Witten had good reason to call this new parameter "the number of colors". He showed that the anomaly implied an electromagnetic interaction which, to agree with experiment, forced the anomaly parameter N_W to be 3, the physical number of colors. But we insist that Witten's anomaly parameter, N_W, is logically distinct from the parameter ($N_{'t} = \infty$) used in defining the $SU(3)/U1$ manifold itself.

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[13] Note the amusing point: Gell-Mann chose the label "color" by analogy to physical color, which has a *basis of three colors*. Gell-Mann's analogy is even more apt in our case, where $N_{'t} \to \infty$ corresponds to the infinite number of colors (distinct wave-lengths) while the number of *basis* colors being three, corresponds to the *modulus* N_W.

[14] Using the same procedure for three-flavors as used by Manohar[4] and by Mattis-Braaten[5] for two flavors.

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